

EFFECT OF THE RELATIVISTIC SPIN ROTATION FOR ONE- AND TWO-PARTICLE SPIN STATES

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Abstract

The effect of the relativistic spin rotation, conditioned by the setting of the spin in the rest frame of a particle and by the noncommutativity of the Lorentz transformations along noncolinear directions, is discussed. In connection with this, the Thomas precession of the spin polarization vector at the motion of a particle along a curvilinear trajectory is considered. The transformations of the correlation tensor components for a system of two spin-1/2 particles at the transition from the c.m.s. of the particle pair to the laboratory frame are investigated. When the particle laboratory velocities are not colinear, the relativistic spin rotation angles for these particles are different. As a result, the relative fractions of the singlet and triplet states in the relativistic system of two free spin-1/2 particles with a nonzero vector of relative momentum depend on the concrete frame in which this two-particle system is analyzed.

1 Introduction

In modern relativistic theory of reactions, the S-matrix is parameterized on the basis of the formalism of inhomogeneous Lorentz group [1-7]. This is similar to the nonrelativistic theory with the essential modification - the relativistic spin rotation. Since, in this formalism, the spin state of a particle is set in its rest frame, the *concrete* description of this state depends on the frame from which the Lorentz transformation to the rest frame is performed. This is the essence of the effect of relativistic spin rotation at the transition from one frame to another [5-7]. The relativistic spin rotation is a purely kinematical effect conditioned by the setting of the spin of a particle in its rest frame and by the additional rotation of the spatial axes at the successive Lorentz transformations along noncolinear directions, the latter leading to the nontransitivity of the parallelism in the theory of relativity (see [8] and references therein).

For two free particles, the total spin results from the addition of the particle spins, defined in the respective rest frames, according to the usual angular momentum addition rules. The operator of the total angular momentum is represented by a sum of the operators of total spin and orbital angular momentum, both commuting with the free Hamiltonian. The price for this quasi nonrelativistic description is the lost relation of the orbital angular momentum operator with the usual field coordinate. Instead, this operator is related with the c.m.s. coordinate introduced by Pryce [9], Newton and Wigner [10] and, within the Dirac theory - by Foldy and Wouthuysen [11] (see also papers [4, 6, 7]).

In the case of a nonzero vector of relative velocity, the angles of the relativistic rotation of the spins of the two particles are different - they depend on the frame from which the Lorentz transformations to the particle rest frames are performed. This leads to the frame dependence of the fractions of the states with different values of the total spin (in particular, the singlet and triplet fractions for a system of two spin-1/2 particles). Thus,

due to the frame dependence of the difference of the spin rotation angles, the square of the total spin of a system of two free particles with a nonzero vector of relative velocity is not a Lorentz invariant.

In Sections 2 and 3 a brief review of the consequences of the relativistic spin rotation for one-particle spin states is given; the connection of the effect of the relativistic spin rotation with the kinematical spatial rotations at successive Lorentz transformations is analyzed. In Section 4, the relation between the relativistic spin rotation and the Thomas precession of the internal angular momentum at the motion of a particle along a curvilinear trajectory is discussed. The precession of the spin polarization vector of a relativistic particle in the magnetic and electric fields is considered in Section 5. In Sections 6 and 7 the effect of the relativistic spin rotation for the states of two free particles with a nonzero vector of relative momentum is considered in detail. The results are summarized in Section 8.

2 Relativistic spatial rotation

Let A be any four-vector set in a frame K : $A = \{A_0, \mathbf{A}\}$, $A^2 = A_0^2 - \mathbf{A}^2$. The components of this four-vector in another frame K' , having parallel respective spatial axes and moving with velocity \mathbf{v} and Lorentz factor $\gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}$ (c is the velocity of light) with respect to the frame K , are given by the Lorentz transformation:

$$\mathbf{A}' = \mathbf{A} - \frac{\mathbf{v}}{c} \frac{\gamma}{\gamma + 1} (A_0 + A'_0), \quad A'_0 = \gamma \left(A_0 - \frac{\mathbf{v}}{c} \mathbf{A} \right), \quad (1)$$

$A'^2 = A^2$. We will denote this transformation as $A' = L(\mathbf{v})A$.¹ Let us further consider another frame K_0 associated with a physical object moving with the velocity \mathbf{v}_0 and Lorentz factor γ_0 , having parallel spatial axes with respect to the frame K . The same object moves with the velocity \mathbf{v}_0' and Lorentz factor γ_0' with respect to the frame K' - we denote \widetilde{K}_0 the associated frame having parallel spatial axes with respect to the frame K' . We will see that, generally, $\widetilde{K}_0 \neq K_0$: the parallel axes of the frames K' and K , K_0 and K do not imply the parallel axes of the frames K_0 and K' . The axes of all the three frames could be mutually parallel if only their velocities were colinear.

Considering a particle of four-velocity $u = p/m$ (p is the particle four-momentum and m is its mass) at rest in the frame \widetilde{K}_0 or K_0 : $u^* = \{c, 0, 0, 0\}$, we can transform it to the frame K' : $u' = L(-\mathbf{v}_0')u^* = \{\gamma_0'c, \gamma_0'\mathbf{v}_0'\}$, and then to the frame K : $u = L(-\mathbf{v})L(-\mathbf{v}_0')u^*$. Comparing the result of the successive transformations with that of the direct one: $u = L(-\mathbf{v}_0)u^* = \{\gamma_0c, \gamma_0\mathbf{v}_0\}$, we get the vector form of the law of the relativistic addition of the velocities \mathbf{v}_0' and \mathbf{v} :

$$\mathbf{v}_0 = \frac{\gamma_0'}{\gamma_0} \left[\mathbf{v}_0' \frac{1}{\gamma} + \mathbf{v} \frac{\gamma_0' + \gamma_0}{\gamma_0'(\gamma + 1)} \right] = \left[\frac{\mathbf{v}_0'}{\gamma} + \mathbf{v} \left(1 + \frac{\mathbf{v} \mathbf{v}_0'}{\mathbf{v}^2} \frac{\gamma - 1}{\gamma} \right) \right] \left(1 + \frac{\mathbf{v} \mathbf{v}_0'}{c^2} \right)^{-1}. \quad (2)$$

¹In accordance with the principle of relativity, the parallelism of the coordinate axes requires that the velocity vector of the frame K with respect to the frame K' is $-\mathbf{v}$. One can check that the inverse transformation $L^{-1}(\mathbf{v}) = L(-\mathbf{v})$, i.e. $A = L(-\mathbf{v})A'$. The above definition of parallelism is not unique in the case when the direction of the frame velocity coincides with one of the coordinate axes. In such a case, the observers in the frames K and K' have to specify the plane containing their relative velocity vector and one of the remaining two axes. This can be easily done, e.g., with the help of a light signal.

The second equality follows from the transformation of the Lorentz factors (see the second equation in Eq. (1)):

$$\gamma_0 = \gamma\gamma'_0 \left(1 + \frac{\mathbf{v}\mathbf{v}'_0}{c^2}\right).$$

At the relativistic addition of velocities in the reverse order: $u = L(-\mathbf{v}'_0)L(-\mathbf{v})u^*$, we obtain

$$\tilde{\mathbf{v}}_0 = \frac{\gamma\gamma'_0}{\gamma_0} \left[\mathbf{v} \frac{1}{\gamma'_0} + \mathbf{v}'_0 \frac{\gamma + \gamma_0}{\gamma(\gamma'_0 + 1)} \right] = \left[\frac{\mathbf{v}}{\gamma'_0} + \mathbf{v}'_0 \left(1 + \frac{\mathbf{v}\mathbf{v}'_0}{\mathbf{v}_0'^2} \frac{\gamma'_0 - 1}{\gamma'_0}\right) \right] \left(1 + \frac{\mathbf{v}\mathbf{v}'_0}{c^2}\right)^{-1}. \quad (3)$$

For colinear vectors \mathbf{v} and \mathbf{v}'_0 , the results of the addition of velocities in the direct and reverse order coincide. In the case of noncolinear velocities \mathbf{v} and \mathbf{v}'_0 , we have $\mathbf{v}_0 \neq \tilde{\mathbf{v}}_0$ (but always $|\mathbf{v}_0| = |\tilde{\mathbf{v}}_0|$, $\gamma_0 = \tilde{\gamma}_0$). Thus, the symmetry in the addition of velocities, in general, is absent. The successive Lorentz transformations along noncolinear directions are noncommutative; the pure Lorentz transformations do not form a group.

Let us now compare the results of direct ($K \rightarrow K_0$) and successive ($K \rightarrow K' \rightarrow \tilde{K}_0$) Lorentz transformations of an arbitrary four-vector A . The direct Lorentz transformation of this four-vector from the frame K to the frame K_0 gives

$$A^* = L(\mathbf{v}_0) A, \quad (4)$$

where \mathbf{v}_0 is the velocity of the frame K_0 in the frame K , determined by the relativistic addition of the velocities \mathbf{v}'_0 and \mathbf{v} . The successive Lorentz transformations from the frame K to the frame K' and then from the frame K' to the frame \tilde{K}_0 lead to the result

$$\tilde{A}^* = L(\mathbf{v}'_0)L(\mathbf{v}) A. \quad (5)$$

It is easy to show (see below) that four-vectors A^* and \tilde{A}^* do not coincide when the velocities \mathbf{v}'_0 and \mathbf{v} are not colinear. Their time components are equal to each other, however the three-vector \mathbf{A}^* is turned with respect to the three-vector $\tilde{\mathbf{A}}^*$ by some angle ω around the vector $[\mathbf{v}'_0\mathbf{v}]$ [12]. Thus,

$$|A^*| = |\tilde{A}^*|, \quad |A_0^*| = |\tilde{A}_0^*|, \quad \mathbf{A}^* = \hat{D}_z(\omega)\tilde{\mathbf{A}}^*, \quad (6)$$

where $\hat{D}_z(\omega)$ is the matrix of an active rotation by the angle ω around the axis z parallel to the vector $[\mathbf{v}'_0\mathbf{v}]$ (the rotation in Eq. (6) recovers the one introduced in Ref. [12] after the substitutions $\hat{D}_z(\omega) \rightarrow \hat{D}_z^{-1}(\omega)$ and $z \rightarrow -z$). This means that spatial axes of the frames \tilde{K}_0 and K_0 are generally not parallel - the transitivity of parallelism is violated in the theory of relativity [8].

To find out the rotation angle ω , let us follow reference [12] and assume that the four-vector A is a four-velocity of the frame K , *i.e.* in this frame $A = \{c, 0, 0, 0\}$. At the direct transition from the frame K to the frame K_0 , we have from Eq. (4):

$$\mathbf{A}^* = -\gamma_2 \mathbf{v}_0 \quad (7)$$

while, at the successive Lorentz transformations $K \rightarrow K' \rightarrow \tilde{K}_0$, we get from Eq. (5):

$$\tilde{\mathbf{A}}^* = -\gamma_0 \tilde{\mathbf{v}}_0. \quad (8)$$

Recall that the velocities \mathbf{v}_0 and $\tilde{\mathbf{v}}_0$ represent the results of relativistic addition of the velocities \mathbf{v}'_0 and \mathbf{v} in direct and reverse order given in Eqs. (2) and (3), respectively. According to Eq. (6),

$$-\mathbf{v}_0 = -\hat{D}_z(\omega)\tilde{\mathbf{v}}_0. \quad (9)$$

Consequently, the angle of the spatial rotation ω is equal to the angle between the velocities \mathbf{v}_0 and $\tilde{\mathbf{v}}_0$. Therefore,

$$\sin \omega = \frac{|[\mathbf{v}_0\tilde{\mathbf{v}}_0]|}{v_0^2}, \quad \cos \omega = \frac{\mathbf{v}_0\tilde{\mathbf{v}}_0}{v_0^2},$$

where $v_0 = |\mathbf{v}_0| = |\tilde{\mathbf{v}}_0|$. The positive sign of the angle ω ($0 \leq \omega \leq \pi$) is fixed by the negative projection of the difference vector $\tilde{\mathbf{v}}_0 - \mathbf{v}_0$ on the direction of the vector \mathbf{v} at nonrelativistic velocities. Using Eqs. (2) and (3), one can show [12] that

$$\sin \omega = \gamma\gamma'_0 \frac{vv'_0}{c^2} \sin \theta \frac{1 + \gamma + \gamma'_0 + \gamma_0}{(1 + \gamma)(1 + \gamma'_0)(1 + \gamma_0)}, \quad (10)$$

where

$$\gamma_0 = \tilde{\gamma}_0 = \gamma\gamma'_0 \left(1 + \frac{vv'_0 \cos \theta}{c^2} \right),$$

$v = |\mathbf{v}|$, $v'_0 = |\mathbf{v}'_0|$, θ is the angle between the vectors \mathbf{v}'_0 and \mathbf{v} ($0 \leq \theta \leq \pi$). Taking into account the equality

$$(1 + \gamma + \gamma'_0 + \gamma_0)^2 = 2(1 + \gamma)(1 + \gamma'_0)(1 + \gamma_0) - (\gamma_0^2 - 1)(\gamma'^2_0 - 1) \sin^2 \theta, \quad (11)$$

one can express the cosine of the spatial rotation angle in the following forms:

$$\cos \omega = \frac{(1 + \gamma + \gamma'_0 + \gamma_0)^2}{(1 + \gamma)(1 + \gamma'_0)(1 + \gamma_0)} - 1 = 1 - \frac{(\gamma - 1)(\gamma'_0 - 1)}{1 + \gamma_0} \sin^2 \theta. \quad (12)$$

Clearly, in the case of colinear velocity vectors \mathbf{v} and \mathbf{v}'_0 ($\sin \theta = 0$), the angle of the relativistic spatial rotation is equal to zero.

3 Relativistic spin rotation

The vector of the spin polarization in some frame is constructed by the Lorentz transformation of the four-tensor of the angular momentum M_{lm} , or the spin four-vector $S_m = (1/2)e_{mlns}M_{lm}u_s$, from this frame into the rest frame of a particle ($e_{mnl s}$ is the antisymmetric unit four-tensor of the fourth range, u is the particle four-velocity). Thus, the spin state of a particle depends on the concrete frame from which the transformation to the rest frame of this particle is performed. As a result, the relativistic spin rotation is a particular case of the kinematical spatial rotation considered above and, it is described by the same relations. It should be noted that this fact was not indicated in the book [12], in which the spatial rotation at successive Lorentz transformations was considered in detail. On the other hand, in papers [5-7] the effect of the relativistic spin rotation was discussed without the connection with the general kinematical effect of the relativistic spatial rotation.

Let us assume that K_0 is the particle rest frame and consider the transformations of the spin four-vector S satisfying the condition $Su = 0$. In the frame K the spin four-vector is S , the particle velocity is \mathbf{v}_0 , and, in accordance with the Lorentz transformation from the frame K to the rest frame K_0 , the spin polarization vector $\boldsymbol{\zeta}$ in the frame K (better to say - related to the frame K) is given by the relation

$$S^* = L(\mathbf{v}_0)S = \{0, \boldsymbol{\zeta}\}. \quad (13)$$

Note that $\boldsymbol{\zeta} = \langle \hat{\mathbf{s}} \rangle$ represents a mean value of the spin operator $\hat{\mathbf{s}}$ in the particle rest frame. It is related to the normalized spin polarization: $\mathbf{P} = \boldsymbol{\zeta}/s$ ($|\mathbf{P}| \leq 1$), where s is the particle spin.

The spin four-vector S' in the frame K' is connected with the spin four-vector S in the frame K by the Lorentz transformation: $S' = L(\mathbf{v})S$, where \mathbf{v} is the velocity of the frame K' with respect to the frame K . In accordance with Eq. (5), we find the spin polarization vector in the frame K' as a result of the transition from the frame K' to the rest frame \widetilde{K}_0 :

$$\widetilde{S}^* = L(\mathbf{v}'_0)L(\mathbf{v})S = \{0, \widetilde{\boldsymbol{\zeta}}\}, \quad (14)$$

where \mathbf{v}'_0 is the particle velocity in the frame K' . According to Eq. (6),

$$\boldsymbol{\zeta} = \hat{D}_z(\omega) \widetilde{\boldsymbol{\zeta}}, \quad (15)$$

$|\boldsymbol{\zeta}| = |\widetilde{\boldsymbol{\zeta}}|$, $\zeta_z = \widetilde{\zeta}_z$ and $z \parallel [\mathbf{v}'_0 \mathbf{v}]$. We see that the angle ω of the relativistic spin rotation around the axis z can be calculated directly with the help of Eqs. (10) and (12) for the kinematical spatial angle. The relation (10) was just given in the Stapp paper [5] on the relativistic theory of polarization phenomena.

Let us consider the dependence of the angle of the relativistic spin rotation $\omega = \omega(\theta)$ on the angle θ between the velocities \mathbf{v}'_0 and \mathbf{v} . Without loss of generality, both angles θ and ω can be considered in the interval $\langle 0, \pi \rangle$. It follows from Eqs. (10) and (12) that $\omega(0) = \omega(\pi) = 0$. The maximal value

$$\omega_{\max} = \omega(\theta_m) = \arccos \left[1 - \frac{2(\gamma - 1)(\gamma'_0 - 1)}{(\gamma + 1)(\gamma'_0 + 1)} \right]$$

is achieved at

$$\theta_m = \arccos \left(-\sqrt{\frac{(\gamma - 1)(\gamma'_0 - 1)}{(\gamma + 1)(\gamma'_0 + 1)}} \right) \geq \frac{\pi}{2}.$$

Note that $\omega_{\max} = \frac{\pi}{2}$, $< \frac{\pi}{2}$ and $> \frac{\pi}{2}$ when $(\gamma'_0 - 1)(\gamma - 1) = 8$, < 8 and > 8 , respectively. At nonrelativistic velocities in the frame K' ($v'_0/c \ll 1$, $\gamma'_0 \approx 1$, $\gamma_0 \approx \gamma$) the angle of the spin rotation is very small ($\omega \ll 1$):

$$\omega \approx \frac{\gamma}{\gamma + 1} \frac{v'_0 v}{c^2} \sin \theta. \quad (16)$$

It follows from Eqs. (10) and (12) that in the ultrarelativistic limit, when $\gamma'_0 \rightarrow \infty$, $\gamma_0/\gamma'_0 \rightarrow \gamma[1 + (v/c) \cos \theta]$:

$$\sin \omega \approx \frac{v}{c} \sin \theta \frac{1 + \gamma[1 + (v/c) \cos \theta]}{(1 + \gamma)[1 + (v/c) \cos \theta]}, \quad \cos \omega \approx 1 - \frac{\gamma - 1}{\gamma[1 + (v/c) \cos \theta]} \sin^2 \theta. \quad (17)$$

At $v/c \approx 1, \gamma \gg 1$ one has

$$\omega_{\max} \approx \pi - 2\sqrt{2/\gamma}, \quad \theta_m \approx \pi - \sqrt{2/\gamma}.$$

In this case the sharp dependence on the angle θ in the vicinity of $\theta = \pi$ with the width $\sim 1/\sqrt{\gamma}$ takes place [7]. Outside this vicinity the angle ω is equal approximately to the angle θ .

The relations (17) are valid exactly for massless particles (photons, neutrinos). In this situation the spin rotation angle coincides with the aberration angle (the angle between the vectors \mathbf{v}'_0 and \mathbf{v}_0); then the spin projection of the particle onto the direction of its momentum (the helicity, the degree of the circular polarization of light) is the relativistic invariant [7].

Indeed, the angle between the particle velocity in the frame K' and the particle velocity in the frame K is determined, in general, by the relations:

$$\begin{aligned} \sin \alpha &= \frac{|[\mathbf{v}'_0 \mathbf{v}_0]|}{v'_0 v_0} = \frac{v}{c} \sin \theta \frac{\gamma}{\gamma + 1} \frac{\gamma'_0 + \gamma_0}{\sqrt{\gamma_0^2 - 1}}, \\ \cos \alpha &= \frac{\mathbf{v}'_0 \mathbf{v}_0}{v'_0 v_0} = \frac{\gamma'_0 \gamma}{\sqrt{\gamma_0^2 - 1}} \left[\frac{v'_0}{c} + \frac{v}{c} \cos \theta - \frac{v'_0}{c} \frac{\gamma - 1}{\gamma} \sin^2 \theta \right]. \end{aligned} \quad (18)$$

One can show that always the angle of the relativistic spin rotation ω at the transformation $K' \rightarrow K$ is less than the angle of the momentum rotation α around the same axis parallel to the vector $[\mathbf{v}'_0 \mathbf{v}]$. Thus, $\omega < \alpha$. However, in the limit

$$v'_0/c \approx 1, \gamma'_0 \gg 1, \gamma_0 = \gamma \gamma'_0 [1 + (v/c) \cos \theta] \gg 1$$

the relations (18) pass into the expressions (17) for massless particles, so then $\omega = \alpha$.

4 Thomas precession and the effect of the relativistic spin rotation

The Thomas precession of the spin polarization vector is the important effect of the relativistic kinematics; it takes place even in the absence of the direct dynamical action on the particle spin [13]. As a typical example of the Thomas precession, one can consider the precession of the spin of a charged particle with zero magnetic moment at its quasi-classical motion in the external magnetic field [14].

Due to the orthogonality condition $S(t)u(t) = 0$ for the spin four-vector $S(t)$ and particle four-velocity $u(t)$ at each instant time t , the following equality holds:

$$dS(t) u(t) + S(t) du(t) = 0. \quad (19)$$

In the absence of a direct dynamical action on the spin, the increment $dS(t)$ can be only a linear combination of the four-vectors $S(t)$ and $u(t)$. Since, in the considered case, the former contribution is forbidden due to the conservation of $S(t)^2$ ($S(t)dS(t) = 0$), the increment $dS(t)$ is proportional to $u(t)$, with the coefficient fixed by the general relation

(19). The spin four-vectors at the instant times t and $t + dt$ are then connected by the relation:

$$dS(t) \equiv S(t + dt) - S(t) = -u(t) (S(t) du(t)). \quad (20)$$

The spin polarization vector is determined in the rest frame of a particle at each instant time; in so doing the time t is given in the laboratory frame K . The Lorentz transformations from the laboratory frame K to the instantaneous rest frames $K_0(t)$ and $K_0(t + dt)$ lead to the equalities

$$\begin{aligned} S^*(t) &= \{0, \boldsymbol{\zeta}(t)\}_{K_0(t)} = L(\mathbf{v}(t)) S(t)_K; \\ S^*(t + dt) &= \{0, \boldsymbol{\zeta}(t + dt)\}_{K_0(t+dt)} = L(\mathbf{v}(t) + d\mathbf{v}(t)) S(t + dt)_K. \end{aligned} \quad (21)$$

Now we will consider the connection between the spin polarization vectors $\boldsymbol{\zeta}(t)$ and $\boldsymbol{\zeta}(t + dt)$. Let us note that, according to Eq. (2) for the relativistic addition of velocities, the velocity of the frame $K_0(t_1)$ with respect to the frame $K_0(t)$, resulting from the addition of the laboratory velocity $\mathbf{v}(t_1)$ of the frame $K_0(t_1)$ and the velocity $-\mathbf{v}(t)$ of the laboratory frame K with respect to the frame $K_0(t)$, is equal to

$$\Delta \mathbf{v}' = \left[\frac{\mathbf{v}(t_1)}{\gamma(t_1)} - \mathbf{v}(t) \left(1 - \frac{\mathbf{v}(t) \mathbf{v}(t_1)}{c^2} \frac{\gamma(t)}{\gamma(t) + 1} \right) \right] \left(1 - \frac{\mathbf{v}(t) \mathbf{v}(t_1)}{c^2} \right)^{-1}. \quad (22)$$

At the infinitesimal time increment one has

$$t_1 = t + dt, \quad \mathbf{v}(t_1) = \mathbf{v}(t) + d\mathbf{v}.$$

Then it follows from Eq. (22) that

$$d\mathbf{v}' = \gamma(t) d\mathbf{v} + \mathbf{v} \frac{d\gamma(t)}{dt} \frac{\gamma^3(t)}{\gamma(t) + 1}. \quad (23)$$

According to the second formula (21), the spin polarization vector of a particle at the instant time $t + dt$ is a result of the direct Lorentz transformation of the spin four-vector at the instant time $t + dt$ from the laboratory frame K to the rest frame of a particle $K_0(t + dt)$. On the other hand, taking into account Eq. (20), one can write

$$L(\mathbf{v}(t)) S(t + dt)_K = \{-S(t) du, \boldsymbol{\zeta}(t)\}_{K_0(t)}.$$

Applying the infinitesimal Lorentz transformation $L(d\mathbf{v}')$ to both sides of this equation, it is easy to see that, in the first order in dt , one arrives at the following equality:

$$L(d\mathbf{v}') L(\mathbf{v}) S(t + dt)_K = \{0, \boldsymbol{\zeta}(t)\}_{\widetilde{K}_0(t+dt)}.$$

Thus, the spin polarization vector at the instant time t can be presented as a result of the successive Lorentz transformations firstly from the laboratory frame to the frame $K_0(t)$ and then, from the frame $K_0(t)$ to the rest frame $\widetilde{K}_0(t + dt)$. In accordance with Eqs. (6) and (15), we obtain

$$\begin{aligned} L(\mathbf{v}(t) + d\mathbf{v}) &= \hat{D}_z(d\omega) L(d\mathbf{v}') L(\mathbf{v}(t)); \\ \boldsymbol{\zeta}(t + dt) &= \hat{D}_z(d\omega) \boldsymbol{\zeta}(t). \end{aligned} \quad (24)$$

In Eq. (24) $\hat{D}_z(d\omega)$ is the matrix of the infinitesimal spatial rotation by the angle $d\omega$ around the instantaneous axis z parallel to the vector $[d\mathbf{v}' \mathbf{v}]$.

We deduce from Eq. (24) that the spin polarization vector rotation for the infinitesimal time interval due the Thomas precession coincides with the relativistic spin rotation at the transition from the frame $K_0(t)$, in which the particle velocity is $d\mathbf{v}'$, to the laboratory frame K , in which the particle velocity is $\mathbf{v}(t) + d\mathbf{v}$.

To find the angular velocity of the Thomas precession one should perform the following substitutions in Eq. (10) for the angle of relativistic spatial rotation:

$$\mathbf{v}'_0 \rightarrow d\mathbf{v}', \gamma'_0 \rightarrow 1, \mathbf{v} \rightarrow \mathbf{v}(t), \gamma \rightarrow \gamma(t), \mathbf{v}_0 \rightarrow \mathbf{v}(t) + d\mathbf{v}, \gamma_0 \rightarrow \gamma(t).$$

Taking into account Eq. (23) for $d\mathbf{v}'$, we obtain the following expression for the infinitesimal angle of the rotation of the spin polarization vector at the particle motion along the curvilinear trajectory (in the vector form):

$$d\boldsymbol{\omega} = \frac{\gamma^2(t)}{\gamma(t) + 1} \frac{1}{c^2} [d\mathbf{v} \mathbf{v}(t)]. \quad (25)$$

Hence the spin polarization vector satisfies the precession equation:

$$\frac{d\boldsymbol{\zeta}(t)}{dt} = [\boldsymbol{\Omega}_{\text{Th}}(t) \boldsymbol{\zeta}(t)] \quad (26)$$

with the angular velocity vector of the Thomas precession [13]:

$$\boldsymbol{\Omega}_{\text{Th}}(t) \equiv \frac{d\boldsymbol{\omega}(t)}{dt} = -\frac{\gamma^2(t)}{\gamma(t) + 1} \frac{1}{c^2} [\mathbf{v}(t) \frac{d\mathbf{v}(t)}{dt}]. \quad (27)$$

It can be presented also as [14]

$$\boldsymbol{\Omega}_{\text{Th}}(t) = -(\gamma(t) - 1)\boldsymbol{\Omega}_0(t), \quad (28)$$

where

$$\boldsymbol{\Omega}_0(t) = \left[\mathbf{l}(t) \frac{d\mathbf{l}(t)}{dt} \right]$$

is the instantaneous angular velocity of the rotation of the particle momentum, $\mathbf{l}(t)$ is the unit vector along the momentum direction. According to Eq. (28), the direction of the angular velocity of the Thomas precession is opposite to the direction of the angular velocity of the momentum rotation. It is clear that in the case of rectilinear motion the Thomas precession of spin is absent.

At nonrelativistic velocities, Eq. (27) reduces to the formula

$$\boldsymbol{\Omega}_{\text{Th}}(t) = -\frac{1}{2c^2} \left[\mathbf{v}(t) \frac{d\mathbf{v}(t)}{dt} \right], \quad (29)$$

which describes, in particular, the Thomas precession of the gyroscope axis.

5 Spin precession in the electromagnetic field

Let us consider the precession of the spin polarization vector of a relativistic particle of charge q , mass m and spin s , moving along a quasiclassical trajectory in the external electromagnetic field. Its magnetic moment

$$\mu = \frac{q\hbar}{2mc}gs, \quad (30)$$

where \hbar is the reduced Planck constant and g is the gyromagnetic ratio determining the anomalous part of the magnetic moment $\mu' = \mu(g - 2)/g$. In particular, in the case of an electron or a negative muon: $q = -|e|$, $s = 1/2$, $g \approx 2 + e^2/(\pi\hbar c)$.

The angular velocity of the spin precession in the electromagnetic field, which is contained in the equation

$$\frac{d\boldsymbol{\zeta}(t)}{dt} = [\boldsymbol{\Omega}(t) \boldsymbol{\zeta}(t)],$$

can be represented as a sum of two terms:

$$\boldsymbol{\Omega}(t) = \boldsymbol{\Omega}_{\text{Th}}(t) + \boldsymbol{\Omega}_{\text{dyn}}(t). \quad (31)$$

Here $\boldsymbol{\Omega}_{\text{Th}}(t)$ is the angular velocity of the Thomas precession, which is described by Eq. (28), and $\boldsymbol{\Omega}_{\text{dyn}}(t)$ is the angular velocity of the dynamical precession conditioned by the presence of the nonzero magnetic field $\mathbf{H}^*(t)$ in the rest frame of a particle. In accordance with the known result of nonrelativistic quantum mechanics,

$$\boldsymbol{\Omega}_{\text{dyn}}(t) = -\frac{1}{\gamma(t)} \frac{qg}{2mc} \mathbf{H}^*(t); \quad (32)$$

the inverse Lorentz factor in the expression (32) appears due to the time contraction in the rest frame in comparison with the laboratory frame ($dt = \gamma(t)dt^*$). According to the Lorentz transformations of the laboratory magnetic and electric fields $\mathbf{H}(t)$ and $\mathbf{E}(t)$ at the point of the particle location, we have

$$\mathbf{H}^*(t) = \gamma(t) \left\{ \mathbf{H}(t) - \frac{\gamma(t) - 1}{\gamma(t)} \mathbf{l}(t)(\mathbf{H}(t)\mathbf{l}(t)) + \left[\mathbf{E}(t) \frac{\mathbf{v}(t)}{c} \right] \right\}, \quad (33)$$

where $\mathbf{l}(t) = \mathbf{v}(t)/v(t)$. The equation of particle motion

$$\frac{d\mathbf{p}(t)}{dt} = q \left(\mathbf{E}(t) + \frac{1}{c} [\mathbf{v}(t) \times \mathbf{H}(t)] \right) \quad (34)$$

determines the instantaneous angular velocity of the rotation of the particle momentum:

$$\boldsymbol{\Omega}_0(t) = -\frac{q}{mc\gamma(t)} \left\{ \mathbf{H}(t) - \mathbf{l}(t)(\mathbf{H}(t)\mathbf{l}(t)) + \frac{\gamma^2(t)}{\gamma^2(t) - 1} \left[\mathbf{E}(t) \frac{\mathbf{v}(t)}{c} \right] \right\}. \quad (35)$$

As a result, using Eqs. (28), (31)-(33) and (35), we arrive at the formula of Bargmann, Michel and Telegdi for the total angular velocity of the spin precession of the relativistic particle at the presence of the external electromagnetic field [15, 16]:

$$\begin{aligned} \boldsymbol{\Omega}(t) = & -\frac{q}{2mc} \left\{ \left(g - 2 + \frac{2}{\gamma(t)} \right) \mathbf{H}(t) - \frac{\gamma(t) - 1}{\gamma(t)} (g - 2) \mathbf{l}(t)(\mathbf{l}(t)\mathbf{H}(t)) \right\} - \\ & -\frac{q}{2mc} \left(g - \frac{2\gamma(t)}{\gamma(t) + 1} \right) \left[\mathbf{E}(t) \frac{\mathbf{v}(t)}{c} \right]. \end{aligned} \quad (36)$$

5.1 Magnetic field

In the case of a constant transverse homogeneous magnetic field, when $\mathbf{E} = 0$, $\mathbf{H}\mathbf{l} = 0$, it follows from Eqs. (35) and (36) that

$$\mathbf{\Omega}_0 = -\frac{q}{mc\gamma}\mathbf{H}; \quad \mathbf{\Omega} = -\frac{q}{mc}\left(\frac{g-2}{2} + \frac{1}{\gamma}\right). \quad (37)$$

Then the angular velocity of the spin precession $\mathbf{\Omega}$ and the angular velocity of the momentum rotation $\mathbf{\Omega}_0$ at the circular motion in the magnetic field are connected by the proportionality relation

$$\mathbf{\Omega} = \left(\frac{g-2}{2}\gamma + 1\right) \mathbf{\Omega}_0. \quad (38)$$

In so doing the angular velocity of the spin rotation with respect to the momentum rotation is determined by the anomalous magnetic moment:

$$\Delta\mathbf{\Omega} = \gamma \frac{g-2}{2} \mathbf{\Omega}_0. \quad (39)$$

The formula (39) forms the theoretical basis for the measurement of the muon anomalous magnetic moment [17]. Indeed, at the Dirac value $g = 2$ the projection of the spin polarization vector onto the momentum direction would not change in time. Let the initial spin polarization vector of a muon $\boldsymbol{\zeta}(t_0)$ be directed along the momentum $\mathbf{p}(t_0)$. After n total turns of the muon in the magnetic field, the angle between the spin polarization vector $\boldsymbol{\zeta}(t_0 + nT)$ and the momentum $\mathbf{p}(t_0 + nT) = \mathbf{p}(t_0)$, where $T = 2\pi/|\mathbf{\Omega}_0|$ is the period of the circular motion, becomes equal to

$$\Delta\theta = n\pi\gamma(g-2). \quad (40)$$

5.2 Electric field

In the absence of the magnetic field ($\mathbf{H} = 0$) it follows from Eqs. (35) and (36) that

$$\mathbf{\Omega}_0(t) = -\frac{q}{mc}\frac{\gamma(t)}{\gamma^2(t)-1}\left[\mathbf{E}(t)\frac{\mathbf{v}(t)}{c}\right], \quad \mathbf{\Omega}(t) = -\frac{q}{2mc}\left(g - \frac{2\gamma(t)}{\gamma(t)+1}\right)\left[\mathbf{E}(t)\frac{\mathbf{v}(t)}{c}\right]. \quad (41)$$

In so doing,

$$\mathbf{\Omega}(t) = \left(\frac{1}{2}(g-2)\frac{\gamma^2(t)-1}{\gamma(t)} + \frac{\gamma(t)-1}{\gamma(t)}\right) \mathbf{\Omega}_0(t). \quad (42)$$

In case of a plane trajectory of a charged particle in the electric field, the vectors $\mathbf{\Omega}(t)$ and $\mathbf{\Omega}_0(t)$ have a constant direction along the normal \mathbf{n} to the plane of the motion. Then the angle of the precession of the spin polarization vector is described by the formula

$$\theta(t) = \int_0^t \left[\frac{1}{2}(g-2)\frac{\gamma^2(t')-1}{\gamma(t')} + \frac{\gamma(t')-1}{\gamma(t')} \right] \frac{d\theta_0(t')}{dt'} dt', \quad (43)$$

where $\theta_0(t)$ is the angle between the initial momentum of the particle and its momentum at the instant time t . In case of a negligible change of the particle kinetic energy at the

motion, the connection between the angle of the spin precession and the angle of the momentum rotation takes a simple form:

$$\theta = \left[\frac{1}{2}(g-2)\frac{\gamma^2-1}{\gamma} + \frac{\gamma-1}{\gamma} \right] \theta_0. \quad (44)$$

In the nonrelativistic case

$$\theta = \frac{1}{2}(g-1)\frac{v^2}{c^2}\theta_0. \quad (45)$$

The relation (43) may be applied to the plane channeling of positively charged particles (protons, positrons) in bent crystals [14]. The distortion of the trajectory of a charged particle at its motion along the bent channel is conditioned by the average electric field in the plane of the channel which is perpendicular to the particle momentum. In so doing, the angle of the particle deviation θ_0 in Eq. (43) coincides with the angle of the bend of the crystal and, the axis of the spin rotation is perpendicular to the plane of the bend of the crystal.

5.3 Thomas precession and the spin-orbit interaction

The central electric field has the structure

$$\mathbf{E} = -\frac{1}{r} \frac{dV(r)}{dr} \mathbf{r}, \quad (46)$$

where $r = |\mathbf{r}|$, $V(r)$ is the central potential. According to Eqs. (32) (33) and (45) the angular velocity of the dynamical spin precession in the central electric field can be presented in the form

$$\boldsymbol{\Omega}_{\text{dyn}} = \frac{qg}{2m^2c^2\gamma} \frac{1}{r} \frac{dV(r)}{dr} \mathbf{L}, \quad (47)$$

where $\mathbf{L} = [\mathbf{r}\mathbf{p}]$ is the orbital angular momentum. On the other hand, it follows from Eqs. (28), (35) and (45) that the angular velocity of the Thomas precession is given by the expression

$$\boldsymbol{\Omega}_{\text{Th}} = -\frac{q}{m^2c^2} \frac{1}{\gamma+1} \frac{1}{r} \frac{dV(r)}{dr} \mathbf{L} \quad (48)$$

As a result, we have:

$$\boldsymbol{\Omega}_{\text{Th}} = -\frac{2}{g} \frac{\gamma}{\gamma+1} \boldsymbol{\Omega}_{\text{dyn}}.$$

For the electron, the gyromagnetic ratio $g \approx 2$ and, in the nonrelativistic approximation ($\gamma \approx 1$), the following relation holds:

$$\boldsymbol{\Omega}_{\text{Th}} = -\frac{1}{2} \boldsymbol{\Omega}_{\text{dyn}}. \quad (49)$$

From the point of view of quantum mechanics, the equality of the spin precession with the angular velocity $\boldsymbol{\Omega}$ is obtained through the commutation of the spin operator $\hat{\mathbf{s}}$ with the interaction Hamiltonian

$$\hat{H}_{\text{int}} = \hbar \hat{\mathbf{s}} \hat{\boldsymbol{\Omega}} = \hbar \hat{\mathbf{s}} \hat{\boldsymbol{\Omega}}_{\text{dyn}} + \hbar \hat{\mathbf{s}} \hat{\boldsymbol{\Omega}}_{\text{Th}}.$$

Taking into account Eq. (49), we arrive at the following expression for the spin-orbit interaction of electrons in atoms [16]:

$$\hat{H}_{\text{int}} = \frac{1}{2} \hbar \hat{\mathbf{s}} \hat{\mathbf{\Omega}}_{\text{dyn}} = -\frac{1}{4} \frac{|e| \hbar}{m^2 c^2} \frac{1}{r} \frac{dV(r)}{dr} \hat{\boldsymbol{\sigma}} \hat{\mathbf{L}}. \quad (50)$$

Here $\hat{\boldsymbol{\sigma}} = 2\hat{\mathbf{s}}$ is the Pauli vector operator, $\hat{\mathbf{L}}$ is the operator of the orbital angular momentum. Eq. (50) contains the additional factor 1/2 due to the contribution of the Thomas precession which reduces twice the effect of the direct interaction of the magnetic moment of moving electron with the electric field in the atom.

The spin-orbit interaction in atoms leads to the fine splitting of atom levels.

6 Effect of the relativistic spin rotation on spin correlations in a two-particle system

The spin correlations in two-particle quantum systems were analyzed in detail as a tool allowing one to measure the space-time characteristics of particle production [18,20-23], to study the two-particle interaction and the production dynamics (see [19,20-22] and references therein) and to verify the consequences of the quantum-mechanical coherence with the help of Bell-type inequalities [21, 22].

The spin state of the system of two particles in an arbitrary frame is described by the two-particle density matrix, the elements of which, $\rho_{m_1 m_1'; m_2 m_2'}^{(1,2)}$, are given in the representation of the spin projections of the first and second particle in the corresponding rest frames onto the common coordinate axis z (see, *e.g.*, [20, 23]). However, one should take into account the relativistic spin rotation conditioned by the additional rotation of the spatial axes at the successive Lorentz transformations along noncolinear directions [5-7]. As a result, the *concrete* description of a particle spin state depends on the frame from which the transition to the particle rest frame is performed. Particularly, the total spin composition of the two-particle state with a nonzero vector of relative momentum is generally frame-dependent due to different relativistic rotation angles of the two spins at the transition to the frame moving in the direction which is not colinear with the velocity vectors of both particles.

Usually, it is convenient to consider the spin correlations in the center-of-mass system (c.m.s.) of the particle pair. This is natural at the addition of the two-particle total spin and the relative orbital angular momentum into the conserved total angular momentum. In some cases, however, it may be useful to make transition to the laboratory, *e.g.*, in the case when the particle scatterings are used as their spin analyzers [18].² Denoting M_1 , M_2 and $\mathbf{p}_1 = \mathbf{k}$, $\mathbf{p}_2 = -\mathbf{k}$ the masses and c.m.s. momenta of the two particles, their respective c.m.s. velocities are $\mathbf{v}_1 = c\mathbf{k}/\sqrt{\mathbf{k}^2 + M_1^2 c^2}$, $\mathbf{v}_2 = -c\mathbf{k}/\sqrt{\mathbf{k}^2 + M_2^2 c^2}$. We denote the corresponding laboratory velocities as \mathbf{v}'_1 and \mathbf{v}'_2 , and - the laboratory velocity of the particle pair as \mathbf{v} . At the Lorentz transformation from the c.m.s. of the particle pair to the laboratory frame with parallel respective spatial axes, the spins of the first and the second particle (in their respective rest frames) rotate in opposite directions around the axis which is parallel to the vector $[\mathbf{k}\mathbf{v}]$. The angles of the relativistic spin rotation of

²In principle, this transition is not necessary since one can transform the four-vectors defining the polarization analyzers first to the pair c.m.s. and then to the respective particle rest frames.

the first ($i = 1$) and second ($i = 2$) particle are found from Eqs. (10) and (12) with the substitutions

$$\mathbf{v}'_0 \rightarrow \mathbf{v}_i, \quad \mathbf{v}_0 \rightarrow \mathbf{v}'_i$$

and similarly for the corresponding Lorentz factors. Thus,

$$\begin{aligned} \sin \omega_1 &= \gamma \gamma_1 \frac{vv_1}{c^2} \sin \theta \frac{1 + \gamma + \gamma_1 + \gamma'_1}{(1 + \gamma)(1 + \gamma_1)(1 + \gamma'_1)}, \\ \cos \omega_1 &= 1 - \frac{(\gamma - 1)(\gamma_1 - 1)}{1 + \gamma'_1} \sin^2 \theta; \end{aligned} \quad (51)$$

$$\begin{aligned} \sin \omega_2 &= -\gamma \gamma_2 \frac{vv_2}{c^2} \sin \theta \frac{1 + \gamma + \gamma_2 + \gamma'_2}{(1 + \gamma)(1 + \gamma_2)(1 + \gamma'_2)}, \\ \cos \omega_2 &= 1 - \frac{(\gamma - 1)(\gamma_2 - 1)}{1 + \gamma'_2} \sin^2 \theta. \end{aligned} \quad (52)$$

Here

$$\gamma'_1 = \gamma \gamma_1 \left(1 + \frac{vv_1}{c^2} \cos \theta\right), \quad \gamma'_2 = \gamma \gamma_2 \left(1 - \frac{vv_2}{c^2} \cos \theta\right); \quad (53)$$

$\theta = \theta_1$ is the angle between the vectors \mathbf{k} (\mathbf{v}_1) and \mathbf{v} ; $v_i = |\mathbf{v}_i|$, $v'_i = |\mathbf{v}'_i|$, $v = |\mathbf{v}|$ and γ_i , γ'_i , γ are the corresponding Lorentz factors. In the case of equal-mass particles the relations $\mathbf{v}_1 = -\mathbf{v}_2$, $\gamma_1 = \gamma_2$ hold (but $\gamma'_1 \neq \gamma'_2$ when $\theta \neq \pi/2$).

It should be noted that the positive sign of the angle of the spin rotation corresponds to the direction of the nearest rotation from the vector \mathbf{k} to the vector \mathbf{v} . It is clear that the angle between the vectors $-\mathbf{k}$ (\mathbf{v}_2) and \mathbf{v} is equal to $\theta_2 = \theta + \pi$, and $\sin \theta_2 = -\sin \theta$, $\cos \theta_2 = -\cos \theta$. Without loss of generality one can assume that $0 \leq \theta \leq \pi$. Since the spins of the first and second particle rotate in opposite directions, the corresponding rotation angles have the opposite signs: $\omega_1 > 0$, $\omega_2 < 0$.

In the case of the colinearity of the vectors \mathbf{k} and \mathbf{v} , when $\theta = 0$ or $\theta = \pi$, both the rotation angles are equal to zero. At nonrelativistic velocities v_i in the c.m.s. of the particle pair the angles ω_i of the relativistic spin rotation are very small and scale with v_i (see Eq. (16) with the substitutions $v'_0 \rightarrow v_1$ and $v'_0 \rightarrow -v_2$).

Taking into account the relativistic spin rotation at the transition from the two-particle c.m.s. to the laboratory, the two-particle spin density matrix is transformed as follows:

$$\hat{\rho}^{(1,2)} = \hat{D}^{(1)}(\omega_1) \otimes \hat{D}^{(2)}(\omega_2) \hat{\rho}^{(1,2)} \hat{D}^{(1)+}(\omega_1) \otimes \hat{D}^{(2)+}(\omega_2), \quad (54)$$

where

$$\hat{D}^{(1)}(\omega_1) = \exp(i\omega_1 \hat{\mathbf{s}}_1 \mathbf{n}), \quad \hat{D}^{(2)}(\omega_2) = \exp(i\omega_2 \hat{\mathbf{s}}_2 \mathbf{n}) \quad (55)$$

are the matrices of the active space rotations generated by the vector spin operators $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$, \mathbf{n} is the unit vector parallel to the direction of the vector $[\mathbf{k}\mathbf{v}]$.

In the case of two spin-1/2 particles, the two-particle spin density matrix has the structure [18,20-22]:

$$\hat{\rho}^{(1,2)} = \frac{1}{4} [\hat{I}^{(1)} \otimes \hat{I}^{(2)} + (\hat{\boldsymbol{\sigma}}^{(1)} \mathbf{P}_1) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes (\hat{\boldsymbol{\sigma}}^{(2)} \mathbf{P}_2) + \sum_{l=1}^3 \sum_{k=1}^3 T_{lk} \hat{\sigma}_l^{(1)} \otimes \hat{\sigma}_k^{(2)}]. \quad (56)$$

Here \hat{I} is the two-row unit matrix, $\hat{\sigma}^{(i)} = 2\hat{s}_i$ are the Pauli vector operators, $\mathbf{P}_i = \langle \hat{\sigma}^{(i)} \rangle$ are the polarization vectors ($i = 1, 2$), $T_{lk} = \langle \hat{\sigma}_l^{(1)} \otimes \hat{\sigma}_k^{(2)} \rangle$ are the components of the correlation tensor, $\{1, 2, 3\} \equiv \{x, y, z\}$. The left and right indexes of the correlation tensor correspond to the rest frames of the first ($i = 1$) and second ($i = 2$) particle, respectively. The corresponding probability to select the particles with the polarizations $\boldsymbol{\alpha}^{(i)}$ can be obtained by the substitution of the matrices $\hat{\sigma}_l^{(i)}$ in the expression (56) with the corresponding projections $\alpha_l^{(i)}$. Particularly, when analyzing the polarization states with the help of particle decays, the vector analyzing power $\boldsymbol{\alpha}^{(i)} = \alpha_i \mathbf{n}_i$, where α_i is the decay asymmetry corresponding to the decay analyzer unit vector \mathbf{n}_i . As a result [20-22], the correlation between the decay analyzers is determined by the product of the decay asymmetries and the trace of the spin correlation tensor

$$T = T_{xx} + T_{yy} + T_{zz}.$$

For example, the angular correlation $\mathbf{n}_1 \mathbf{n}_2 = \cos \theta_{12}$ between the directions of the three-momenta of the decay protons in the respective rest frames of two Λ -hyperons decaying into the channel $\Lambda \rightarrow p + \pi^-$ with the P -odd asymmetry $\alpha = 0.642$ is described by the normalized probability density [19-22]

$$W(\cos \theta_{12}) = \frac{1}{2} \left(1 + \alpha^2 \frac{T}{3} \cos \theta_{12} \right). \quad (57)$$

Clearly, the structure of both Eq. (56) and the corresponding angular distribution of the spin analyzers (e.g., Eq. (57)) does not depend on the system from which the transitions to the particle rest frames are performed. The system dependence manifests only through the relativistic rotations in the successive Lorentz transformations along noncolinear directions. This circumstance was not understood in paper [19], where the unnecessary condition of nonrelativistic velocities of Λ -particles was required.

7 Transformations of the spin states of two spin-1/2 particles

The matrices of the space rotations due to the transition from the c.m.s. of two free spin-1/2 particles to the laboratory are the following:

$$\hat{D}^{(1)}(\omega_1) = \cos \frac{\omega_1}{2} + i \hat{\sigma}^{(1)} \mathbf{n} \sin \frac{\omega_1}{2}, \quad \hat{D}^{(2)}(\omega_2) = \cos \frac{\omega_2}{2} + i \hat{\sigma}^{(2)} \mathbf{n} \sin \frac{\omega_2}{2}. \quad (58)$$

Selecting the z -axis parallel to the direction of the vector $\mathbf{n} = [\mathbf{k}\mathbf{v}]/|[\mathbf{k}\mathbf{v}]|$, and the axes x and y in the plane perpendicular to this vector, the polarization vectors and the spin correlation tensor transform at the transition to the laboratory in accordance with the active rotations around the z -axis by the angles ω_1 and ω_2 for the first and second particle, respectively (the components of the polarization vectors transform according to Eq. (15) with the substitutions: $\tilde{\zeta} \rightarrow \mathbf{P}_i/2$ and $\zeta \rightarrow \mathbf{P}'_i/2$):

$$\begin{aligned} P'_{i;x} &= P_{i;x} \cos \omega_i - P_{i;y} \sin \omega_i; & P'_{i;y} &= P_{i;y} \cos \omega_i + P_{i;x} \sin \omega_i; & P'_{i;z} &= P_{i;z} \\ T'_{xx} &= (T_{xx} \cos \omega_1 - T_{yx} \sin \omega_1) \cos \omega_2 - (T_{xy} \cos \omega_1 - T_{yy} \sin \omega_1) \sin \omega_2; \end{aligned}$$

$$\begin{aligned}
T'_{yy} &= (T_{yy} \cos \omega_1 + T_{xy} \sin \omega_1) \cos \omega_2 + (T_{yx} \cos \omega_1 + T_{xx} \sin \omega_1) \sin \omega_2; & T'_{zz} &= T_{zz}; \\
T'_{xy} &= (T_{xy} \cos \omega_1 - T_{yy} \sin \omega_1) \cos \omega_2 + (T_{xx} \cos \omega_1 - T_{yx} \sin \omega_1) \sin \omega_2; \\
T'_{yx} &= (T_{yx} \cos \omega_1 + T_{xx} \sin \omega_1) \cos \omega_2 - (T_{yy} \cos \omega_1 + T_{xy} \sin \omega_1) \sin \omega_2; \\
T'_{xz} &= T_{xz} \cos \omega_1 - T_{yz} \sin \omega_1; & T'_{zx} &= T_{zx} \cos \omega_2 - T_{zy} \sin \omega_2; \\
T'_{yz} &= T_{yz} \cos \omega_1 + T_{xz} \sin \omega_1; & T'_{zy} &= T_{zy} \cos \omega_2 + T_{zx} \sin \omega_2.
\end{aligned} \tag{59}$$

Particularly, the trace of the spin correlation tensor transforms at the transition to the laboratory as:

$$T' = (T_{xx} + T_{yy}) \cos(\omega_1 - \omega_2) + (T_{xy} - T_{yx}) \sin(\omega_1 - \omega_2) + T_{zz} \tag{60}$$

or, in the case of a symmetric tensor, as:

$$T' = T - 2(T_{xx} + T_{yy}) \sin^2 \frac{\omega_1 - \omega_2}{2}. \tag{61}$$

So, the c.m.s. trace T in Eq. (57) is substituted by the laboratory one T' calculated using Eq. (60) or (61) together with Eqs. (10) and (12) for the spin rotation angles.

It was shown [20-22] (see also [24, 25]) that the trace of the correlation tensor of a system of two spin-1/2 particles is the following linear combination of the relative fractions of singlet (the total spin $S = 0$) and triplet ($S = 1$) states:

$$T = \langle \hat{\sigma}^{(1)} \otimes \hat{\sigma}^{(2)} \rangle = \rho_t - 3\rho_s, \quad \rho_t + \rho_s = 1. \tag{62}$$

When we have the pure singlet state of the particle pair in its c.m.s. ($\rho_s = 1$, $\rho_t = 0$, $T_{lk} = -\delta_{lk}$, $T = -3$), the transformation to the laboratory gives

$$T' = -3 + 4 \sin^2 \frac{\omega_1 - \omega_2}{2}. \tag{63}$$

It follows from Eqs. (62) and (63) that at the transition to the laboratory the relative fraction of the singlet state decreases in favor of a triplet state:

$$\rho'_s = \cos^2 \frac{\omega_1 - \omega_2}{2}, \quad \rho'_t = \sin^2 \frac{\omega_1 - \omega_2}{2}. \tag{64}$$

Thus the square of the total spin of two free particles with a nonzero vector of relative velocity is not a relativistic invariant (see [6]). Introducing the two-particle singlet state:

$$|\psi\rangle_{00} = \frac{1}{\sqrt{2}} \left(| +1/2\rangle_z^{(1)} | -1/2\rangle_z^{(2)} - | -1/2\rangle_z^{(1)} | +1/2\rangle_z^{(2)} \right) \tag{65}$$

and the triplet state with the zero projection onto the rotation axis z :

$$|\psi\rangle_{10} = \frac{1}{\sqrt{2}} \left(| +1/2\rangle_z^{(1)} | -1/2\rangle_z^{(2)} + | -1/2\rangle_z^{(1)} | +1/2\rangle_z^{(2)} \right), \tag{66}$$

the result in Eq. (64) also follows directly from the matrices of space rotations in Eq. (58); the singlet state in the two-particle c.m.s. is transformed into the following superposition of the singlet and triplet states in the laboratory:

$$|\psi'_s\rangle = \cos \frac{\omega_1 - \omega_2}{2} |\psi\rangle_{00} + i \sin \frac{\omega_1 - \omega_2}{2} |\psi\rangle_{10}. \tag{67}$$

Similarly, the transformation of the pure triplet state $|\psi\rangle_{10}$ in the two-particle c.m.s. ($\rho_s = 0, \rho_t = 1, T_{zz} = -1, T_{xx} = T_{yy} = 1, T = 1$) to the laboratory gives

$$T' = 1 - 4 \sin^2 \frac{\omega_1 - \omega_2}{2}, \quad (68)$$

the corresponding fractions being

$$\rho'_s = \sin^2 \frac{\omega_1 - \omega_2}{2}, \quad \rho'_t = \cos^2 \frac{\omega_1 - \omega_2}{2}, \quad (69)$$

in accordance with the transformation:

$$|\psi'_t\rangle = \cos \frac{\omega_1 - \omega_2}{2} |\psi\rangle_{10} + i \sin \frac{\omega_1 - \omega_2}{2} |\psi\rangle_{00}. \quad (70)$$

In the case of the unpolarized triplet in the two-particle c.m.s. ($\rho_s = 0, \rho_t = 1, T_{lk} = (1/3)\delta_{lk}, T = 1$ [18, 22]) we have

$$T' = 1 - \frac{4}{3} \sin^2 \frac{\omega_1 - \omega_2}{2}, \quad (71)$$

$$\rho'_s = \frac{1}{3} \sin^2 \frac{\omega_1 - \omega_2}{2}, \quad \rho'_t = 1 - \frac{1}{3} \sin^2 \frac{\omega_1 - \omega_2}{2}. \quad (72)$$

Using Eqs. (10) and (12), it is easy to show that in the case of two spin-1/2 particles with the same masses ($\gamma_2 = \gamma_1, v_2 = v_1$) the measure of the spin mixing,

$$\kappa = \sin^2 \frac{\omega_1 - \omega_2}{2},$$

can be written in the form:

$$\kappa = \frac{(v_1 v)^2}{c^2} \sin^2 \theta \left[\left(\frac{1}{\gamma} + \frac{1}{\gamma_1} \right)^2 + \frac{(v_1 v)^2}{c^2} \sin^2 \theta \right]^{-1}. \quad (73)$$

The maximum of the mixing factor κ corresponds to the angle $\theta = \pi/2$. In the ultrarelativistic limit, when $\gamma_1 \gg 1, \gamma \gg 1$ and $\sin \theta \gg \max(1/\gamma, 1/\gamma_1)$, the factor κ approaches unity. Then the singlet state in the two-particle c.m.s. becomes in the laboratory the triplet state with the zero projection of the total spin onto the spin rotation axis z and, *vice versa*.

8 Summary

1. The effect of the relativistic spin rotation at the transition from some frame to another frame is analyzed. The essence of this effect lies in the fact that in the framework of the formalism of inhomogeneous Lorentz group the particle spin state is set in the particle rest frame and, its concrete description depends on the frame from which the Lorentz transformation to the rest frame is performed.

2. The connection between the relativistic spin rotation and the relativistic spatial rotation at the successive Lorentz transformations along noncolinear directions is considered. It is shown that the relativistic spin rotation is a purely kinematical effect; the

rotation angle coincides with the angle between the resulting velocities at the relativistic addition of two velocities in the direct and reverse order.

3. The Thomas precession of the spin polarization vector at the particle motion along a curvilinear trajectory is studied. It is shown that Thomas precession is the effect of the relativistic kinematics connected with the effect of the relativistic spin rotation; the expression for the angular velocity of the Thomas precession is derived using the formula for the angle of the relativistic spin rotation.

4. The spin precession of a relativistic charged particle in the external electromagnetic field is considered. The cases of the precession in the homogeneous magnetic field and in the electric field are discussed. It is established that the angular velocity of the spin precession can be presented as a simple sum of the angular velocities of the dynamical precession due to the direct interaction of the particle magnetic moment with the magnetic field in the instantaneous rest frame and, the kinematical Thomas precession connected with the motion of a charge particle under the action of the Lorentz force.

5. The effect of the relativistic spin rotation for a system of two free particles is investigated setting their spin states in the corresponding particle rest frames. The transition from the c.m.s. of two particles to the laboratory frame is considered. It is shown that the angles of the relativistic spin rotation of two particles are generally different, except for the case of colinear vectors of particle velocities in the laboratory.

6. As a result, the effect of the relativistic spin rotation leads to the dependence of the total two-particle spin composition (the singlet and triplet fractions in the particular case of spin-1/2 particles) on the *concrete* frame in which the two-particle system with a nonzero vector of relative velocity is analyzed; the total spin is not a Lorentz invariant. The physical origin of this dependence is the violation of the parallelism of the spatial axes of the particle rest frames, except for the case when the Lorentz transformations to these frames are done along the directions colinear with the relative velocity (e.g., from the c.m.s. of the two particles).

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